HEAT TRANSFER

CONDUCTION

The Conduction Rate Equation

– Developed from observed phenomena rather than from first principles.

– Consider a cylindrical rod of know material and it is insulated on its lateral surface, while its ends are maintained at different temperatures.

– The temperature difference causes conduction heat transfer in the positive x direction.

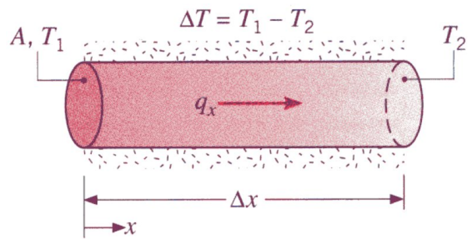


Figure 1: Cylindrical rod

– We are able to measure the heat transfer rate, q_x, and seek to determine how q_x depends on the following variables:

- The temperature difference, ΔT
- The length of the rod, Δx
- The cross-sectional area, A

– By holding ΔT and Δx constant we find A is proportional to q_x.

– By holding ΔT and A constant we find Δx is inversely proportional to q_x.

– By holding A and Δx constant we find ΔT is proportional to q_x.

– Hence collectively: $q_x \propto A \dfrac{\Delta T}{\Delta x}$

– The above proportionality is still valid for different materials (i.e. metal to plastic). However, for same values of ΔT, Δx and A, the value of q_x is smaller for plastic than for metal.

– Therefore the proportionality may be converted to an equality by introducing a coefficient that is a measure of the material behavior. Hence:

$$q_x = kA \frac{\Delta T}{\Delta x} \qquad (W)$$

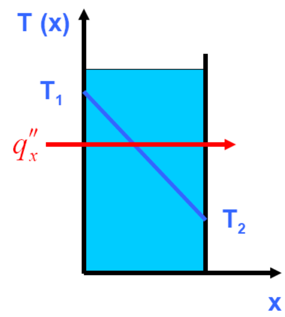
where k is the thermal conductivity (W / m•K).

– Evaluating the expression in the limit as Δx approaches 0, we obtain for heat rate:

$$q_x = -kA\frac{dT}{dx} \qquad (W)$$

– The heat flux is then:

$$q_x'' = \frac{q_x}{A} = -k\frac{dT}{dx} \qquad (W/m^2)$$



Thermal Conductivity

– The thermal conductivity associated with Fourier's law in the x direction is defined as:

$$k_x \equiv -\frac{q_x''}{(\partial T/\partial x)} \qquad (W/m \cdot K)$$

It is similar for thermal conductivities in the y and z directions (k_y and k_z).

– However, for isotropic material the thermal conductivity is independent of the direction i.e. $k_x = k_y = k_z$

– In general, $k_{solid} > k_{liquid} > k_{gas}$

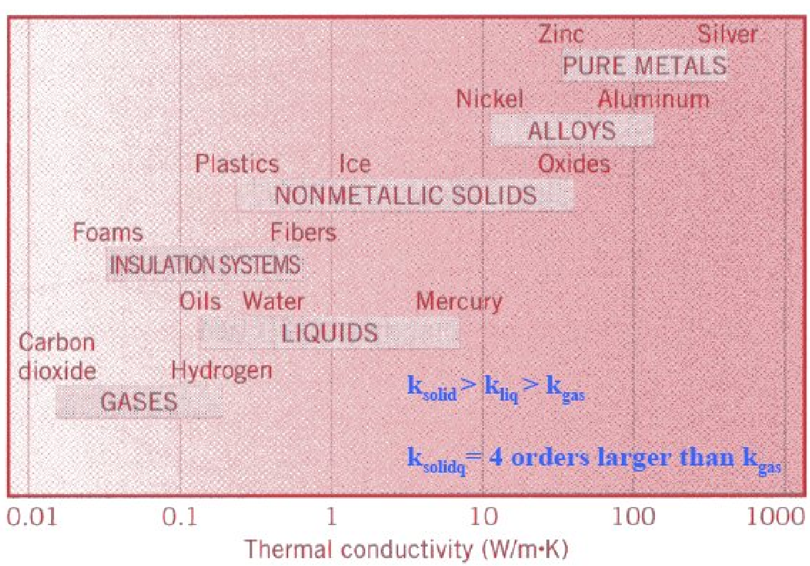
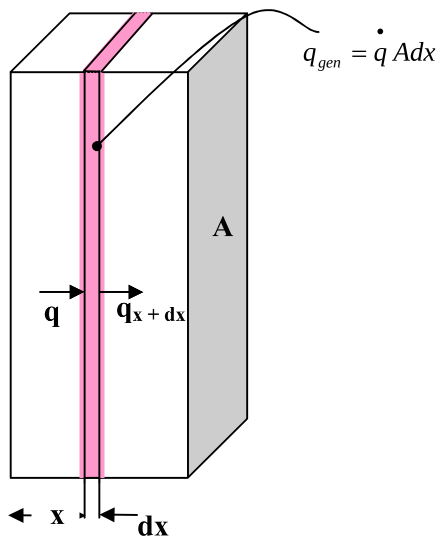


Figure 2: Range of thermal conductivity for various states of matter at normal temperatures and pressures

Heat Diffusion Equation

- Also known as heat equation.
- This is the basic equation that governs the transfer of heat in a solid.
- Basic tool for determining the temperature distribution of a system i.e. T (x, y, z)
- Consider the system shown in the figure below undergoing transient heat conduction with energy generation within the system.



- For the element thickness dx, the energy balance is as such:

Energy conducted in left face + Heat generated within the element - Energy conducted out right face = Change in internal energy

Or

$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st}$$

Energy conducted in left face = $q_x = -kA\frac{\partial T}{\partial x}$

Heat generated within the element = $\dot{q} \quad Adx$

Energy conducted out right face = $q_{x+dx} = \left[-kA\frac{\partial T}{\partial x}\right]_{x+dx}$

$$= -A\left[k\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx\right]$$

Energy conducted out right face $= \rho CA\frac{\partial T}{\partial t}dx$

Where: $\dot{q}$ = energy generated per unit volume, W/m3

C = specific heat capacity of material, J/kg.°C

ρ = density, kg/m3

Hence:

$$-kA\frac{\partial T}{\partial x} + \dot{q}\,Adx + A\left[k\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx\right] = \rho CA\frac{\partial T}{\partial t}dx$$

$$-k\frac{\partial T}{\partial x} + \dot{q}\,dx + k\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx = \rho C\frac{\partial T}{\partial t}dx$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{q} = \rho C\frac{\partial T}{\partial t}$$

Heat equation (three dimensional conduction):

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho C\frac{\partial T}{\partial t}$$

One dimensional heat equation:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\dot{q}=\rho C\frac{\partial T}{\partial t}$$

Simplification of the heat equation:

1. Steady-state condition:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\dot{q}=0$$

2. Steady-state, one dimensional heat transfer with no energy generation:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0$$

Radial coordinates heat equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho C\frac{\partial T}{\partial t}$$

One dimensional radial heat equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\dot{q}=\rho C\frac{\partial T}{\partial t}$$

Simplification of the radial heat equation:

1. Steady-state condition:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\dot{q}=0$$

2. Steady-state, one dimensional heat transfer with no energy generation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)=0$$

Spherical coordinates heat equation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(k\frac{\partial T}{\partial\phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial z}\left(k\sin\theta\frac{\partial T}{\partial z}\right)+\dot{q}=\rho C\frac{\partial T}{\partial t}$$

One dimensional spherical heat equation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\dot{q}=\rho C\frac{\partial T}{\partial t}$$

Simplification of the spherical heat equation:

1. Steady-state condition:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\dot{q}=0$$

2. Steady-state, one dimensional heat transfer with no energy generation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)=0$$

Boundary and Initial Conditions

- To determine the temperature distribution, we have to solve the appropriate form of the heat equation.
- To this we must specify the boundary conditions (the physical conditions existing at the boundaries) and the initial condition of the medium if it is time dependent.
- Since the heat equation is second order in the spatial coordinate, two boundary conditions must be specified and since it is first order in time only one initial condition required.
- The three kinds of boundary conditions are…

1. Constant surface temperature:

$$T(0,t) = T_s$$

Dirichlet condition

2. Constant surface heat flux:

(a) Finite heat flux:

$$-k \left.\frac{\partial T}{\partial x}\right|_{x=0} = q_s''$$

Neumann condition

(b) Adiabatic or insulated surface:

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

3. Convection surface condition:

$$-k \left.\frac{\partial T}{\partial x}\right|_{x=0} = h\left[T_\infty - T(0,t)\right]$$